



## 30 Matrizen

### Aufgabe 30.1 (Multiplikation von Matrizen):

Berechnen Sie für die Matrizen

$$A = \begin{pmatrix} 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 3 \\ 1 & 2 \\ 7 & -5 \end{pmatrix}$$

alle möglichen Matrizenprodukte.

### Lösung von Aufgabe 30.1:

$$A \cdot B = \begin{pmatrix} 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 13$$

$$A \cdot C = \begin{pmatrix} 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 17 & 22 & 27 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$$

$$C \cdot D = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 2 \\ 7 & -5 \end{pmatrix} = \begin{pmatrix} 27 & -8 \\ 75 & -8 \end{pmatrix}$$

$$D \cdot B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \\ 7 & -5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 17 \\ 8 \\ -1 \end{pmatrix}$$

$$D \cdot C = \begin{pmatrix} 4 & 3 \\ 1 & 2 \\ 7 & -5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 19 & 26 & 33 \\ 11 & 14 & 17 \\ -18 & -16 & -14 \end{pmatrix}$$

**Aufgabe 30.2a):**

Lösen Sie die Matrixgleichung  $\begin{pmatrix} 3 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 3 & -2 \end{pmatrix} \cdot X = E$ , wobei  $E$  die entsprechende Einheitsmatrix ist.

**Lösung von Aufgabe 30.2a)**

|      |                                            |                                                                | $\Sigma$                                                                                               | Regie          |
|------|--------------------------------------------|----------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|----------------|
| (1)  | $\begin{pmatrix} 3 & 3 & 2 \end{pmatrix}$  | $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$                      | 9                                                                                                      | 1              |
| (2)  | $\begin{pmatrix} 2 & 2 & 1 \end{pmatrix}$  | $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$                      | 6                                                                                                      | 1              |
| (3)  | $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$                      | 3                                                                                                      | 1 -2 -3        |
| (4)  | $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$                      | 3                                                                                                      | 2              |
| (5)  | $\begin{pmatrix} 0 & -6 & 8 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & -3 \end{pmatrix}$                     | 0                                                                                                      | 1 1 -2         |
| (6)  | $\begin{pmatrix} 0 & -4 & 5 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & -2 \end{pmatrix}$                     | 0                                                                                                      | 3              |
| (7)  | $\begin{pmatrix} 2 & 0 & 4 \end{pmatrix}$  | $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$                     | 6                                                                                                      | 1              |
| (8)  | $\begin{pmatrix} 0 & -6 & 8 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & -3 \end{pmatrix}$                     | 0                                                                                                      | 1              |
| (9)  | $\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} -2 & 3 & 0 \end{pmatrix}$                     | 0                                                                                                      | 4 8 -1         |
| (10) | $\begin{pmatrix} 2 & 0 & 0 \end{pmatrix}$  | $\begin{pmatrix} -7 & 12 & -1 \end{pmatrix}$                   | 6                                                                                                      | $\frac{1}{2}$  |
| (11) | $\begin{pmatrix} 0 & -6 & 0 \end{pmatrix}$ | $\begin{pmatrix} -15 & 24 & -3 \end{pmatrix}$                  | 0                                                                                                      | $-\frac{1}{6}$ |
| (12) | $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$  | $\begin{pmatrix} 2 & -3 & 0 \end{pmatrix}$                     | 0                                                                                                      | 1              |
| (13) | $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$  | $\frac{1}{2} \cdot \begin{pmatrix} -7 & 12 & -1 \end{pmatrix}$ | also $X = \frac{1}{2} \cdot \begin{pmatrix} -7 & 12 & -1 \\ -5 & -8 & 1 \\ 4 & -6 & 0 \end{pmatrix}$ . |                |
| (14) | $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$  |                                                                |                                                                                                        |                |
| (15) | $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$  |                                                                |                                                                                                        |                |

**Aufgabe 30.2b):**

Lösen Sie die Matrixengleichung  $\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 4 & 2 \\ 6 & 0 \end{pmatrix}$

**Lösung von Aufgabe 30.2b)**

|      |                                                 |                                                 |                                                  |                                                 |    |     | $\Sigma$ | Regie                 |
|------|-------------------------------------------------|-------------------------------------------------|--------------------------------------------------|-------------------------------------------------|----|-----|----------|-----------------------|
| (1)  | <span style="border: 1px solid black;">1</span> | 1                                               | -1                                               | -1                                              | 1  | 1   | 2        | 1 -2 -2 -3            |
| (2)  | 2                                               | 5                                               | -7                                               | -5                                              | -2 | 0   | -7       | 1                     |
| (3)  | 2                                               | -1                                              | 1                                                | 3                                               | 4  | 2   | 11       | 1                     |
| (4)  | 5                                               | 2                                               | -4                                               | 3                                               | 6  | 0   | 12       | 1                     |
| (5)  | 1                                               | 1                                               | -1                                               | -1                                              | 1  | 1   | 2        |                       |
| (6)  | 0                                               | <span style="border: 1px solid black;">3</span> | -5                                               | -3                                              | -4 | -2  | -11      | 1 1 1 1               |
| (7)  | 0                                               | -3                                              | 3                                                | 5                                               | 2  | 0   | 7        | 1                     |
| (8)  | 0                                               | -3                                              | 1                                                | 8                                               | 1  | -5  | 2        | 1                     |
| (9)  | 3                                               | 0                                               | 2                                                | 0                                               | 7  | 5   | 17       | 1                     |
| (10) | 0                                               | 3                                               | -5                                               | -3                                              | -4 | -2  | -11      | 2                     |
| (11) | 0                                               | 0                                               | <span style="border: 1px solid black;">-2</span> | 2                                               | -2 | -2  | -4       | 1 -5 $\frac{1}{2}$ -2 |
| (12) | 0                                               | 0                                               | -4                                               | 5                                               | -3 | -7  | -9       | 1                     |
| (13) | 3                                               | 0                                               | 0                                                | 2                                               | 5  | 3   | 13       | 1                     |
| (14) | 0                                               | 6                                               | 0                                                | -16                                             | 2  | 6   | -2       | 1                     |
| (15) | 0                                               | 0                                               | -1                                               | 1                                               | -1 | -1  | -2       | 1                     |
| (16) | 0                                               | 0                                               | 0                                                | <span style="border: 1px solid black;">1</span> | 1  | -3  | -1       | -2 16 -1 1            |
| (17) | 3                                               | 0                                               | 0                                                | 0                                               | 3  | 9   | 15       |                       |
| (18) | 0                                               | 6                                               | 0                                                | 0                                               | 18 | -42 | -18      |                       |
| (19) | 0                                               | 0                                               | -1                                               | 0                                               | -2 | 2   | -1       |                       |
| (20) | 0                                               | 0                                               | 0                                                | 1                                               | 1  | -3  | -1       |                       |

also  $X = \begin{pmatrix} 1 & 3 \\ 3 & -7 \\ 2 & -2 \\ 1 & -3 \end{pmatrix}$ .

denn aus der letzten Zeile (Gleichung (20)) ergibt sich

**(21)**  $x_{14} = 1$  und  $x_{24} = -3$

Gleichung (19) lautet

**(22)**  $-1 \cdot x_{13} = -2$ , also  $x_{13} = 2$  und  $-1 \cdot x_{23} = 2$ , also  $x_{23} = -2$ .

Gleichung (18) lautet

**(23)**  $6 \cdot x_{12} = 18$ , also  $x_{12} = 3$  und  $6 \cdot x_{22} = 6$ , also  $x_{22} = -7$ .

Gleichung (17) lautet

**(24)**  $3 \cdot x_{11} = 3$ , also  $x_{11} = 1$  und  $3 \cdot x_{21} = 9$ , also  $x_{21} = 3$ .

**Aufgabe 30.3:**

Invertieren Sie - sofern möglich - die Matrizen

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 & -2 & -3 \\ -5 & 1 & 2 \\ -3 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -2 & -1 \\ 3 & 7 & 4 \\ -2 & 17 & 9 \end{pmatrix}$$

**Lösung von Aufgabe 30.3**Matrix A:

|      |          |          |           |    |    |    |   |   |    |    |
|------|----------|----------|-----------|----|----|----|---|---|----|----|
| (1)  | <u>1</u> | 0        | 2         | 1  | 0  | 0  | 4 | 1 | -2 | -1 |
| (2)  | 2        | 1        | 4         | 0  | 1  | 0  | 8 |   | 1  |    |
| (3)  | 1        | 2        | 1         | 0  | 0  | 1  | 5 |   |    | 1  |
| (4)  | 1        | 0        | 2         | 1  | 0  | 0  | 4 |   |    |    |
| (5)  | 0        | <u>1</u> | 0         | -2 | 1  | 0  | 0 |   |    |    |
| (6)  | 0        | 2        | -1        | -1 | 0  | 1  | 0 |   |    |    |
| (7)  | 1        | 0        | 2         | 1  | 0  | 0  | 4 |   |    |    |
| (8)  | 0        | 1        | 0         | -2 | 1  | 0  | 0 |   |    |    |
| (9)  | 0        | 0        | <u>-1</u> | 3  | -2 | 1  | 0 |   |    |    |
| (10) | 1        | 0        | 0         | 7  | -4 | 2  | 4 |   |    |    |
| (11) | 0        | 1        | 0         | -2 | 1  | 0  | 0 |   |    |    |
| (12) | 0        | 0        | 1         | -3 | 2  | -1 | 0 |   |    |    |

Also ist  $A^{-1} = \begin{pmatrix} 7 & -4 & 2 \\ -2 & 1 & 0 \\ -3 & 2 & -1 \end{pmatrix}.$

Matrix B:

|      |          |           |           |     |     |     |     |                |                |               |
|------|----------|-----------|-----------|-----|-----|-----|-----|----------------|----------------|---------------|
| (1)  | <u>7</u> | -2        | -3        | 1   | 0   | 0   | 3   | 1              | 5              | 3             |
| (2)  | -5       | 1         | 2         | 0   | 1   | 0   | -1  |                | 7              |               |
| (3)  | -3       | 1         | 1         | 0   | 0   | 1   | 0   |                |                | 7             |
| (4)  | 7        | -2        | -3        | 1   | 0   | 0   | 3   | 3              |                |               |
| (5)  | 0        | <u>-3</u> | 0         | 5   | 7   | 0   | 8   | -2             | -1             | 1             |
| (6)  | 0        | 1         | -2        | 3   | 0   | 7   | 9   |                |                | 3             |
| (7)  | 21       | 0         | -7        | -7  | -14 | 0   | -7  | 1              |                |               |
| (8)  | 0        | 3         | 1         | -5  | -7  | 0   | -8  |                | 7              |               |
| (9)  | 0        | 0         | <u>-7</u> | 14  | 7   | 21  | 35  | -1             | 1              | -1            |
| (10) | 21       | 0         | 0         | -21 | -21 | -21 | -42 | $\frac{1}{21}$ | 0              | 0             |
| (11) | 0        | 21        | 0         | -21 | -42 | 21  | -21 | 0              | $\frac{1}{21}$ | 0             |
| (12) | 0        | 0         | 7         | -14 | -7  | -21 | -35 | 0              | 0              | $\frac{1}{7}$ |
| (13) | 1        | 0         | 0         | -1  | -1  | -1  |     |                |                |               |
| (14) | 0        | 1         | 0         | -1  | -2  | 1   |     |                |                |               |
| (15) | 0        | 0         | 1         | -2  | -1  | -3  |     |                |                |               |

Also ist  $B^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2 & 1 \\ -2 & -1 & -3 \end{pmatrix}.$

Matrix  $C$ :

|     |                                                 |                                                  |    |    |    |   |    |    |    |    |
|-----|-------------------------------------------------|--------------------------------------------------|----|----|----|---|----|----|----|----|
| (1) | <span style="border: 1px solid black;">1</span> | -2                                               | -1 | 1  | 0  | 0 | -1 | 1  | -3 | 2  |
| (2) | 3                                               | 7                                                | 4  | 0  | 1  | 0 | 15 |    | 1  |    |
| (3) | -2                                              | 17                                               | 9  | 0  | 0  | 1 | 25 |    |    | 1  |
| (4) | 1                                               | -2                                               | -1 | 1  | 0  | 0 | -1 | 13 |    |    |
| (5) | 0                                               | <span style="border: 1px solid black;">13</span> | 7  | -3 | 1  | 0 | 18 | 2  | 1  | -1 |
| (6) | 0                                               | 13                                               | 7  | 2  | 0  | 1 | 23 |    |    | 1  |
| (7) | 13                                              | 0                                                | 1  | 7  | 2  | 0 | 23 |    |    |    |
| (8) | 0                                               | 13                                               | 7  | -3 | 1  | 0 | 18 |    |    |    |
| (9) | 0                                               | 0                                                | 0  | 5  | -1 | 1 | 5  |    |    |    |

Also ist  $C$  nicht invertierbar.

**Aufgabe 30.4:**

Invertieren Sie die folgenden „Hilbert-Matrizen“ (die inversen Matrizen enthalten nur ganzzahlige Koeffizienten):

| Aufgabe                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | Lösung                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
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| $A_1 = \begin{pmatrix} 1 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $A_1^{-1} = \begin{pmatrix} 1 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| $A_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | <table><tr><td></td><td></td><td></td><td><math>\Sigma</math></td><td>Regie</td></tr><tr><td>(1)</td><td><math>\begin{pmatrix} 1 &amp; \frac{1}{2} \\ \frac{1}{2} &amp; \frac{1}{3} \end{pmatrix}</math></td><td><math>\begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math></td><td><math>\frac{5}{2}</math></td><td>2</td></tr><tr><td>(2)</td><td></td><td></td><td><math>\frac{11}{6}</math></td><td>6</td></tr><tr><td>(3)</td><td><math>\begin{pmatrix} \boxed{2} &amp; 1 \\ 3 &amp; 2 \end{pmatrix}</math></td><td><math>\begin{pmatrix} 2 &amp; 0 \\ 0 &amp; 6 \end{pmatrix}</math></td><td>5</td><td>1 -3</td></tr><tr><td>(4)</td><td></td><td></td><td>11</td><td>2</td></tr><tr><td>(5)</td><td><math>\begin{pmatrix} 2 &amp; 1 \\ 0 &amp; \boxed{1} \end{pmatrix}</math></td><td><math>\begin{pmatrix} 2 &amp; 0 \\ -6 &amp; 12 \end{pmatrix}</math></td><td>5</td><td>1</td></tr><tr><td>(6)</td><td></td><td></td><td>7</td><td>-1 1</td></tr><tr><td>(7)</td><td><math>\begin{pmatrix} 2 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math></td><td><math>\begin{pmatrix} 8 &amp; -12 \\ -6 &amp; 12 \end{pmatrix}</math></td><td>-2</td><td><math>\frac{1}{2}</math></td></tr><tr><td>(8)</td><td></td><td></td><td>7</td><td>1</td></tr><tr><td>(9)</td><td><math>\begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math></td><td><math>\begin{pmatrix} 4 &amp; -6 \\ -6 &amp; 12 \end{pmatrix}</math></td><td colspan="2"><math>A_2^{-1} = \begin{pmatrix} 4 &amp; -6 \\ -6 &amp; 12 \end{pmatrix}</math></td></tr><tr><td>(10)</td><td></td><td></td><td></td><td></td></tr></table> |                                                                              |                                                              |                | $\Sigma$      | Regie | (1) | $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\frac{5}{2}$ | 2 | (2) |  |  | $\frac{11}{6}$ | 6 | (3) | $\begin{pmatrix} \boxed{2} & 1 \\ 3 & 2 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$ | 5 | 1 -3 | (4) |  |  | 11 | 2 | (5) | $\begin{pmatrix} 2 & 1 \\ 0 & \boxed{1} \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 \\ -6 & 12 \end{pmatrix}$ | 5 | 1 | (6) |  |  | 7 | -1 1 | (7) | $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 8 & -12 \\ -6 & 12 \end{pmatrix}$ | -2 | $\frac{1}{2}$ | (8) |  |  | 7 | 1 | (9) | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}$ | $A_2^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}$ |  | (10) |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                              |                                                              | $\Sigma$       | Regie         |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (1)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$               | $\frac{5}{2}$  | 2             |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (2)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                              |                                                              | $\frac{11}{6}$ | 6             |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (3)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $\begin{pmatrix} \boxed{2} & 1 \\ 3 & 2 \end{pmatrix}$                       | $\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$               | 5              | 1 -3          |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (4)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                              |                                                              | 11             | 2             |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (5)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $\begin{pmatrix} 2 & 1 \\ 0 & \boxed{1} \end{pmatrix}$                       | $\begin{pmatrix} 2 & 0 \\ -6 & 12 \end{pmatrix}$             | 5              | 1             |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (6)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                              |                                                              | 7              | -1 1          |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (7)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$                               | $\begin{pmatrix} 8 & -12 \\ -6 & 12 \end{pmatrix}$           | -2             | $\frac{1}{2}$ |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | (8)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                              |                                                              | 7              | 1             |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| (9)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | $\begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}$                            | $A_2^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}$ |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| (10)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| $A_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                 | $A_3^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| $A_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$                                                                                                                                                                                                                                                                                              | $A_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| $A_5 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{pmatrix}$                                                                                                                                                               | $A_5^{-1} = \begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |
| $A_6 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \end{pmatrix}$ | $A_6^{-1} = \begin{pmatrix} 36 & -630 & 3360 & -7560 & 7560 & -2772 \\ -630 & 14700 & -88200 & 211680 & -220500 & 83160 \\ 3360 & -88200 & 564480 & -1411200 & 1512000 & -582120 \\ -7560 & 211680 & -1411200 & 3628800 & -3969000 & 1552320 \\ 7560 & -220500 & 1512000 & -3969000 & 4410000 & -1746360 \\ -2772 & 83160 & -582120 & 1552320 & -1746360 & 698544 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                              |                                                              |                |               |       |     |                                                                              |                                                |               |   |     |  |  |                |   |     |                                                        |                                                |   |      |     |  |  |    |   |     |                                                        |                                                  |   |   |     |  |  |   |      |     |                                                |                                                    |    |               |     |  |  |   |   |     |                                                |                                                   |                                                              |  |      |  |  |  |  |